SOME BOUNDARY VALUE PROBLEMS OF MATHEMATICAL PHYSICS AND OF THE THEORY OF ELASTICITY FOR A HYPERBOLOID OF REVOLUTION OF ONE SHEET

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PMM Vol. 30, No. 5, 1966, pp. 889-896

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(Received December 7, 1965)

The authors investigate the technique of solving boundary value problems of the potential and elasticity theory for hyperboloids of revolution of one sheet. The technique is based on the application of integral representation related to the well known Mellor-Fock expression.

Introduction. Harmonic boundary value problems for hyperboloids of revolution and problems of the theory of elasticity reducible to the former, belong to the class of problems with separable variables and can be solved without any fundamental difficulties.

Nevertheless, the only case investigated in detail seems to be that of the hyperboloids of two sheets [1 to 4], while corresponding problems referring to the regions bounded by the surface of hyperboloid of one sheet were, in fact, not considered at all. That was apparently caused by the lack of a sufficiently developed mathematical method for solving the problems of this type. In such a method, major role would be played by the theorem on expansion of an arbitrary function $f(x)$ defined on the interval $(-\infty, \infty)$ into an integral in terms of spherical functions

$$f(x)=\int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh \pi\tau}\left\{\frac{P_{-1/2+i\tau}(ix)+P_{-1/2+i\tau}(-ix)}{2}\int_{-\infty}^{\infty} f(y)\,\frac{P_{-1/2+i\tau}(iy)+P_{-1/2+i\tau}(-iy)}{2}\,dy\,+\right.$$
$$\left.+\,\frac{P_{-1/2+i\tau}(ix)-P_{-1/2+i\tau}(-ix)}{2i}\int_{-\infty}^{\infty} f(y)\,\frac{P_{-1/2+i\tau}(iy)-P_{-1/2+i\tau}(-iy)}{2i}\,dy\right\}d\tau \qquad (0.1)$$
$$(-\infty<x<\infty)$$

which was proved by the authors a short time ago in [5].*

* The theorem is valid for functions satisfying the conditions:

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Present work utilises this theorem to obtain solutions of some problems of mathematical physics and of the theory of elasticity. The results obtained can be of use in investigating a wide class of boundary value problems for hyperboloids of one sheet.

1. Dirichlet problem for a hyperboloid of revolution of one sheet. Let (α, β, φ) be a system of curvilinear coordinates connected with the system of cylindrical coordinates (r, z, φ), by

$$r = c \cosh \alpha \sin \beta, \quad z = c \sinh \alpha \cos \beta \qquad (-\infty < \alpha < \infty,\ 0 \leqslant \beta \leqslant {}^1/_2 \pi) \tag{1.1}$$

Laplace's equation in this coordinate system will be

$$\frac{1}{\cosh \alpha} \frac{\partial}{\partial \alpha} \cosh \alpha \frac{\partial u}{\partial \alpha} + \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \sin \beta \frac{\partial u}{\partial \beta} + \left(\frac{1}{\cosh^2 \beta} - \frac{1}{\operatorname{ch}^2 \alpha} \right) \frac{\partial^2 u}{\partial \varphi^2} = 0 \tag{1.2}$$

and it will possess an infinity of particular solutions*

$$u = u_\tau(\alpha, \beta) = [A(\tau) P_{-1/2+i\tau}(i \sinh \alpha) + B(\tau) P_{-1/2+i\tau}(-i \sinh \alpha)] \times$$
$$\times [C(\tau) P_{-1/2+i\tau}(\cos \beta) + D(\tau) P_{-1/2+i\tau}(-\cos \beta)] \qquad (0 \leqslant \tau < \infty) \tag{1.3}$$

where $P_\nu(z)$ is a spherical Legendre function of the first kind.

When we consider the inner Dirichlet problem, we must choose the solutions bounded on the symmetry axis $r = 0$. These solutions will be

$$u = u_\tau(\alpha, \beta) = [M(\tau) P_{-1/2+i\tau}(i \sinh \alpha) + N(\tau) P_{-1/2+i\tau}(-i \sinh \alpha)] P_{-1/2+i\tau}(\cos \beta)$$
$$(-\infty < \alpha < \infty,\ 0 \leqslant \beta < {}^1/_2 \pi,\ 0 \leqslant \tau < \infty) \tag{1.4}$$

In case of the outer Dirichlet problem, respective solutions will be selected with help of the condition of boundedness of grad u on the line $r = c$, $z = 0$ and will be of the type

$$u = u_\tau(\alpha, \beta) =$$
$$= M(\tau) [P_{-1/2+i\tau}(i \sinh \alpha) + P_{-1/2+i\tau}(-i \sinh \alpha)][P_{-1/2+i\tau}(\cos \beta) + P_{-1/2+i\tau}(-\cos \beta)] +$$
$$+ N(\tau) [P_{-1/2+i\tau}(i \sinh \alpha) - P_{-1/2+i\tau}(-i \sinh \alpha)] [P_{-1/2+i\tau}(\cos \beta) - P_{-1/2+i\tau}(-\cos \beta)]$$
$$(-\infty < \alpha < \infty,\ 0 < \beta \leqslant {}^1/_2 \pi,\ 0 \leqslant \tau < \infty) \tag{1.5}$$

Inner Dirichlet problem for a hyperboloid of revolution $\beta = \beta_0$ (see figure) can be formulated as follows. We require to find a function u harmonic in $0 \leqslant \beta \leqslant \beta_0$, continuous in the closed region $0 \leqslant \beta \leqslant \beta_0$, satisfying the boundary conditions

$$u = f(\sinh \alpha) \quad \text{when } \beta = \beta_0 \tag{1.6}$$

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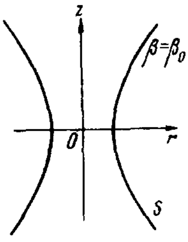
(1). Function $f(x)$ is continuous and has a bounded variation on the open interval $(-\infty, \infty)$ (with exception perhaps, of arbitrarily small neighborhood of a finite number of points.

(2).

$$f(x) |x|^{-1/2} \ln(1 + |x|) \in L(-\infty, -a), \qquad f(x) \in L(-a, a)$$
$$f(x) x^{-1/2} \ln(1 + x) \in L(a, \infty), \qquad a > 0$$

* Later we shall consider the case when u is independent of φ.



and the conditions at infinity

$$u \to 0 \quad \text{as} \quad |\alpha| \to \infty \tag{1.7}$$

uniformly in β. We assume that the given function $f(\sinh\alpha)$ is continuous on the open interval $(-\infty, \infty)$ and tends to zero as $|\alpha| \to \infty$.

In the following we shall restric ourselves to the case when $f(\sinh\alpha)$ is an even function* and we shall seek a solution in the form

$$u = 2\int_0^\infty \frac{\tau \tanh\pi\tau}{\cosh\pi\tau} F(\tau)\, \frac{P_{-1/2+i\tau}(\cos\beta)}{P_{-1/2+i\tau}(\cos\beta_0)}\, \frac{P_{-1/2-i\tau}(i\sinh\alpha) + P_{-1/2+i\tau}(-i\sinh\alpha)}{2}\, d\tau \tag{1.8}$$

$$(-\infty < \alpha < \infty,\ 0 \leqslant \beta < \beta_0)$$

where $F(\tau)$ is a coefficient to be determined.

Inserting (1.8) into the boundary condition (1.6), we find

$$f(\sinh\alpha) = 2\int_0^\infty \frac{\tanh\pi\tau}{\cosh\pi\tau} F(\tau)\, \frac{P_{-1/2+i\tau}(i\sinh\alpha) + P_{-1/2+i\tau}(-i(\sinh\alpha)}{2}\, d\tau \qquad (-\infty < \alpha < \infty) \tag{1.9}$$

from which, using the transformation formula following from (0.1), we obtain

$$F(\tau) = \int_0^\infty f(\sinh\alpha)\, \frac{P_{-1/2+i\tau}(i\sinh\alpha) + P_{-1/2+i\tau}(-i\sinh\alpha)}{2} \cosh\alpha\, d\alpha \tag{1.10}$$

Equations (1.8) and (1.10) together, constitute a formal solution to our problem. To prove our result rigorously, we shall assume that the given function $f(\sinh\alpha)$ can be represented in the form of the integral (1.9), with $F(\tau)$ defined by (1.10) and

$$\sqrt{\tau}\, e^{-1/2\pi\tau} F(\tau) \in L(0, \infty) \tag{1.11}$$

Then, from the estimates

$$\frac{P_{-1/2+i\tau}(\cos\beta)}{P_{-1/2+i\tau}(\cos\beta_0)} \leqslant 1 \qquad (0 \leqslant \beta \leqslant \beta_0) \tag{1.12}$$

$$\left| \frac{P_{-1/2+i\tau}(-i\sinh\alpha) + P_{-1/2+i\tau}(-i\sinh\alpha)}{2} \right| \leqslant \left(\frac{\sinh\pi\tau}{\pi\tau}\right)^{1/2} \frac{P_{-1/2}(i\sinh\alpha) + P_{-1/2}(-i\sinh\alpha)}{2} \tag{1.13}$$

$$(-\infty < \alpha < \infty)$$

which follow from the properties of spherical function, we have

$$2\int_0^\infty \left| \frac{\tau\tanh\pi\tau}{\cosh\pi\tau} F(\tau)\, \frac{P_{-1/2+i\tau}(\cos\beta)}{P_{-1/2+i\tau}(\cos\beta_0)}\, \frac{P_{-1/2+i\tau}(i\sinh\alpha) + P_{-1/2+i\tau}(-i\,\mathrm{sh}\,\alpha)}{2} \right| d\tau \leqslant \tag{1.14}$$

$$\leqslant \frac{2\sqrt{2}}{\sqrt{\pi}}\, \frac{P_{-1/2}(i\sinh\alpha) + P_{-1/2}(-i\sinh\alpha)}{2} \int_0^\infty \sqrt{\tau}\, e^{-1/2\pi\tau} |F(\tau)|\, d\tau$$

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* The case when $f(\sinh\alpha)$ is an odd function of α can be dealt with in the analogous manner.

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i.e. the integral (1.8) converges absolutely.

In the last estimate, the α-dependent factor is bounded for any α, hence convergence of the integral (1.8) is uniform on any closed region D $(-A \leqslant \alpha \leqslant A,\ 0 \leqslant \beta \leqslant \beta_0)$, where A is an arbitrarily large constant. Taking into account the fact that the integrand is harmonic in D and using Harnack Theorem we conclude, that u is harmonic in D and, since A is arbitrary, is also harmonic within the hyperboloid $\beta = \beta_0$.

Uniform convergence also implies the possibility of a limiting process $\beta \to \beta_0$ under the integral sign in (1.8). Therefore, by (1.9), we have

$$u\,|_{\beta=\beta_0} = 2\int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh \pi\tau} F(\tau)\, \frac{P_{-1/2+i\tau}(i \sinh \alpha) + P_{-1/2+i\tau}(-i \sinh \alpha)}{2}\, d\tau = f(\sinh \alpha)$$

Finally, from (1.14) it follows that

$$u = O(1)\, \frac{P_{-1/2}(i \sinh \alpha) + P_{-1/2}(-i \sinh \alpha)}{2} \to 0, \qquad |\alpha| \to \infty$$

uniformly in β. Thus we find, that function u defined by (1.8), satisfies all the required conditions and is a solution of Dirichlet problem.

2. Electrostatic problem on the distribution of charges induced on the surface of a conducting hyperboloid. As a second example we shall compute the field due to a point charge q situated at the point $r = z = 0$ on the axis of a hollow conducting hyperboloid. Representing the potential as

$$u = \frac{q}{\sqrt{r^2 + z^2}} - u_1 \tag{2.1}$$

we can reduce the problem of determining u_1 to finding the solution of the Dirichlet problem discussed above, with the boundary condition

$$u_1\,|_{\beta=\beta_0} = f(\sinh \alpha) = \frac{q}{c\sqrt{\sinh^2 \alpha + \sin^2 \beta_0}} \tag{2.2}$$

Function $f(\sinh \alpha)$ satisfies the conditions of the expansion theorem (0.1), consequently it can be represented in the integral form (1.9). After necessary calculations, we obtain

$$\begin{aligned} F(\tau) &= \frac{q}{c}\int_0^\infty \frac{1}{\sqrt{\sinh^2 \alpha + \sin^2 \beta_0}}\, \frac{P_{-1/2+i\tau}(i \sinh \alpha) + P_{-1/2+i\tau}(-i \sinh \alpha)}{2} \cosh \alpha\, d\alpha = \\ &= \frac{\pi q}{c}\, \frac{P_{-1/2+i\tau}(0)}{\cosh \pi\tau}\, \frac{P_{-1/2+i\tau}(\cos \beta_0) + P_{-1/2+i\tau}(-\cos \beta_0)}{2} \end{aligned} \tag{2.3}$$

and*

$$\begin{aligned} \frac{q}{c\sqrt{\sinh^2 \alpha + \sin^2 \beta_0}} &= \frac{\pi q}{c}\int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh^2 \pi\tau} P_{-1/2+i\tau}(0) \times \\ &\times [P_{-1/2+i\tau}(\cos \beta_0) + P_{-1/2+i\tau}(-\cos \beta_0)]\, \frac{P_{-1/2+i\tau}(i \sinh \alpha) + P_{-1/2+i\tau}(-i \sinh \alpha)}{2}\, d\tau \\ &(-\infty < \alpha < \infty) \end{aligned} \tag{2.4}$$

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In the general case, the problem is first separated into two parts, symmetric and antisymmetric.

* Since the derivation of (2.3) is fairly involved, we present here its final form.

Asymptotic behavior of spherical functions as $\tau \to \infty$ implies that

$$\sqrt{\tau}\, e^{-1/2\pi\tau} F(\tau) = O(\tau^{-1/2} e^{-\beta_0 \tau})$$

Consequently, condition (1.11) is in this case fulfilled.

Using (1.8) we find

$$u_1 = \frac{\pi q}{c} \int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh^2 \pi\tau} P_{-1/2+i\tau}(0)\, [P_{-1/2+i\tau}(\cos\beta_0) + P_{-1/2+i\tau}(-\cos\beta_0)] \times$$
$$\times \frac{P_{-1/2+i\tau}(\cos\beta)}{P_{-1/2+i\tau}(\cos\beta_0)} \frac{P_{-1/2+i\tau}(i \sinh\alpha) + P_{-1/2+i\tau}(-i\sinh\alpha)}{2} d\tau \qquad \begin{pmatrix} -\infty < \alpha < \infty \\ 0 \leqslant \beta \leqslant \beta_0 \end{pmatrix} \tag{2.5}$$

Representing the potential of the initial field in form of the integral (1.8) where β_0 is replaced with the arbitrary β $(0 < \beta \leqslant {}^1/_2\,\pi)$, we can obtain an expansion analogous to (2.5) for the final potential u

$$u = \frac{\pi q}{c} \int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh^2 \pi\tau} P_{-1/2+i\tau}(0) \times$$
$$\times \frac{P_{-1/2+i\tau}(\cos\beta_0)\, P_{-1/2+i\tau}(-\cos\beta) - P_{-1/2+i\tau}(-\cos\beta_0)\, P_{-1/2+i\tau}(\cos\beta)}{P_{-1/2+i\tau}(\cos\beta_0)} \times$$
$$\times \frac{P_{-1/2+i\tau}(i\sinh\alpha) + P_{-1/2+i\tau}(-i\sinh\alpha)}{2} d\tau \qquad \begin{pmatrix} -\infty < \alpha < \infty \\ 0 < \beta \leqslant \beta_0 \end{pmatrix} \tag{2.6}$$

Formulas (2.5) and (2.6) together constitute the solution of our problem.

Density of induced charges distributed on the inner surface of the hyperboloid can be found from the formula

$$\sigma = \frac{1}{4\pi c} \frac{1}{\sqrt{\cosh^2\alpha - \sin^2\beta_0}} \frac{\partial u}{\partial \beta}\bigg|_{\beta=\beta_0} \tag{2.7}$$

Inserting (2.6) into it, we obtain

$$\sigma = -\frac{q}{2\pi c^2 \sin\beta_0} \frac{1}{\sqrt{\cosh^2\alpha - \sin^2\beta_0}} \int_0^\infty \frac{\tau \tanh \pi\tau}{\cosh \pi\tau} \times$$
$$\times \frac{P_{-1/2+i\tau}(0)}{P_{-1/2+i\tau}(\cos\beta_0)} \frac{P_{-1/2+i\tau}(i\sinh\alpha) + P_{-1/2+i\tau}(-i\sinh\alpha)}{2} d\tau \tag{2.8}$$
$$(-\infty < \alpha < \infty)$$

In the particular case when $\beta_0 = {}^1/_2\,\pi$, the hyperboloid degenerates into a plane $z = 0$ with a circular opening of radius c, and the integral (2.8) can be expressed in terms of elementary functions. After few operations we arrive at

$$\sigma = -\frac{q}{2\pi^2 c^2 |\sinh\alpha| \cosh^2\alpha} = -\frac{qc}{2\pi^2 r^2 \sqrt{r^2 - c^2}} \tag{2.9}$$

3. Problem of torsion of a shaft which has the form of a hyperboloid of revolution. Let us consider the problem of torsion of a shaft which has a form of a hyperboloid of revolution and the surface S of which is acted upon by external forces, whose distribution density is

$$\mathbf{p} = p\,(N)\cdot \mathbf{i}\ , \qquad N \in S \tag{3.1}$$

In this case, components of the displacement vector will be given by

$$u = 0, \qquad v = v\,(r,\, z), \qquad w = 0$$

where v satisfies

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} - \frac{1}{r^2} v = 0 \tag{3.2}$$

together with the boundary conditions

$$r \frac{d}{dn}\frac{1}{r} v \bigg|_S = \frac{p}{G} \tag{3.3}$$

where $\mathbf{n}$ is the outer normal to S, and G is the torsional modulus (see eg. [6]). Function $p = p\,(N)$ should satisfy the condition

$$\iint\limits_S r p\,(N)\, ds = 0 \tag{3.4}$$

i.e. total sum of the moments of forces acting on the shaft should be equal to zero.

Having defined v, we can find components of the stress tensor, using the formulas

$$\begin{gathered} \sigma_r = \sigma_\varphi = \sigma_z = 0, \\ \tau_{r\varphi} = Gr\frac{\partial}{\partial r}\frac{1}{r} v, \quad \tau_{\varphi z} = G\frac{\partial v}{\partial z}\ , \quad \tau_{zr} = 0 \end{gathered} \tag{3.5}$$

Introducing curvilinear coordinates $(\alpha\,,\,\beta,\varphi)$ and using (1.1), we obtain

$$\frac{1}{\cosh\alpha}\frac{\partial}{\partial\alpha}\cosh\alpha\,\frac{\partial v}{\partial\alpha} + \frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\sin\beta\,\frac{\partial v}{\partial\beta} - \left(\frac{1}{\sin^2\beta} - \frac{1}{\cosh^2\alpha}\right) v = 0 \tag{3.6}$$

$$\frac{\partial}{\partial\beta}\frac{1}{\sin\beta}\, v \bigg|_{\beta=\beta_0} = \frac{c\,\sqrt{\cosh^2\alpha - \sin^2\beta_0}}{\sin\beta_0}\,\frac{p\,(\alpha)}{G} \tag{3.7}$$

In the following we shall assume that $p\,(\alpha)$ is an odd function of α diminishing sufficiently fast at infinity. Equality (3.4) is then fulfilled and the torsional problem can be reduced to finding a solution of (3.6) regular in the region $0 \leqslant \beta \leqslant \beta_0$, satisfying the boundary condition (3.7) and tending to zero at infinity.

Using Fourier's method, we shall seek this solution in the form

$$v = \int_0^\infty M\,(\tau)\,P^1_{-1/2+i\tau}\,(\cos\beta)\,\frac{P^1_{-1/2+i\tau}\,(i\sinh\alpha) + P^1_{-1/2+i\tau}\,(-i\sinh\alpha)}{2}\,d\tau \tag{3.8}$$

where $P^m_\nu(z)$ is the associated Legendre function*.

Inserting (3.8) into (3.7), we obtain

$$c\,\sqrt{\cosh^2\alpha - \sin^2\beta_0}\,\frac{p\,(\alpha)}{G} = \int_0^\infty M\,(\tau)\,P^2_{-1/2+i\tau}\,(\cos\beta_0)\,\frac{P^1_{-1/2+i\tau}\,(i\sinh\alpha) + P^1_{-1/2+i\tau}\,(-i\sinh\alpha)}{2}\,d\tau \tag{3.9}$$

$$(-\infty < \alpha < \infty)$$

* It is easily seen, that the integrand in (3.8) is a particular solution of (3.6) regular in $0 < \beta < \beta_0$ and odd in α.

which can be written as

$$\frac{c}{G}\frac{d}{d\alpha}\int_{-\infty}^{\alpha}\sqrt{\cosh^2\alpha'\sin^2\beta_0}\,p(\alpha')\,d\alpha' =$$

$$= \frac{d}{d\alpha}\int_0^\infty M(\tau)\,P^2_{-1/2+i\tau}(\cos\beta_0)\,\frac{P_{-1/2+i\tau}(i\sinh\alpha)+P_{-1/2+i\tau}(-i\sinh\alpha)}{2}\,d\tau \tag{3.10}$$

and will obviously be fulfilled, if $M(\tau)$ is chosen from the condition

$$f(\sinh\alpha) = \frac{c}{G}\int_{-\infty}^{\alpha}\sqrt{\cosh^2\alpha'-\sin^2\beta_0}\,p(\alpha')\,d\alpha' =$$

$$= \int_0^\infty M(\tau)\,P^2_{-1/2+i\tau}(\cos\beta_0)\,\frac{P_{-1/2+i\tau}(i\sinh\alpha)+P_{-1/2+i\tau}(-i\sinh\alpha)}{2}\,d\tau \tag{3.11}$$

$$(-\infty<\alpha<\infty)$$

Using the theorem (0.1) we find, that

$$M(\tau)=\frac{2\tau\tanh\pi\tau}{\cosh\pi\tau}\,\frac{1}{P^2_{-1/2+i\tau}(\cos\beta_0)}\int_0^\infty f(\sinh\alpha)\,\frac{P_{-1/2+i\tau}(i\sinh\alpha)+P_{-1/2+i\tau}(-i\sinh\alpha)}{2}\cosh\delta\, d\alpha \tag{3.12}$$

Let us now apply the above formulas to the solution of the problem of torsion of a shaft by two concentrated moments $\pm M_0$ applied along the circumference $\alpha = \pm\alpha_0$, $\beta=\beta_0$. In this case wehave

$$p(\alpha) = \pm\, p_0\delta(\alpha\mp\alpha_0) \qquad (\alpha \gtrless 0)$$

$$p_0 = \frac{M_0}{2\pi c^3\cosh^2\alpha_0\sin^2\beta_0\sqrt{\cosh^2\alpha_0-\sin^2\beta_0}} \tag{3.13}$$

where $\delta(x)$ is the delta function. Corresponding expression for $f(\sinh\alpha)$ will be

$$f(\mathrm{sh}\,\alpha) = -\frac{M_0}{2\pi Gc^2\cosh^2\alpha_0\sin^2\beta_0} \qquad (-\alpha_0<\alpha<\alpha_0) \tag{3.14}$$

while outside the given interval, $f(\sinh\alpha)=0$.

Putting (3.14) into (3.12) and taking

$$\int_0^{\alpha_0}\frac{P_{-1/2+i\tau}(i\sinh\alpha)+P_{-1/2+i\tau}(-i\sinh\alpha)}{2}\cosh\alpha\,d\alpha$$

$$= -\frac{1}{1/4+\tau^2}\cosh\alpha_0\,\frac{P^1_{-1/2+i\tau}(i\sinh\alpha_0)+P^1_{-1/2+i\tau}(-i\sinh\alpha_0)}{2} = \tag{3.15}$$

into account, we obtain

$$M(\tau) = \frac{M_0}{\pi Gc^2\cosh\alpha_0\sin^2\beta_0}\,\frac{\tau}{1/4+\tau^2}\,\frac{\tanh\pi\tau}{\cosh\pi\tau}\times$$

$$\times\frac{1}{P^2_{-1/2+i\tau}(\cos\beta_0)}\,\frac{P^1_{-1/2+i\tau}(i\sinh\alpha_0)+P^1_{-1/2+i\tau}(-i\sinh\alpha_0)}{2} \tag{3.16}$$

from which

$$v = \frac{M_0}{\pi G c^2 \cosh\alpha_0 \sin^2\beta_0} \int_0^\infty \frac{\tau \tanh \pi\tau}{(1/4+\tau^2)\cosh\pi\tau} \frac{P^1_{-1/2+i\tau}(\cos\beta)}{P^2_{-1/2+i\tau}(\cos\beta_0)} \times$$

$$\times \frac{P^1_{-1/2+i\tau}(i\sinh\alpha_0) + P^1_{-1/2+i\tau}(-i\sinh\alpha_0)}{2} \frac{P^1_{-1/2+i\tau}(i\sinh\alpha) + P^1_{-1/2+i\tau}(-i\sinh\alpha)}{2} d\tau \quad (3.17)$$

$$(-\infty < \alpha < \infty, \ 0 \leqslant \beta \leqslant \beta_0)$$

follows.

It can easily be checked that the formal solution (3.17) satisfies all conditions of the problem. Let us, in addition, compute the tangential stresses in the cross-section $z = 0$. From (3.5) and (3.17), we have

$$\tau_{\varphi z}\,|_{z=0} = -\frac{M_0}{\pi c^3 \cosh\alpha_0 \sin^2\beta_0 \cos\beta} \int_0^\infty \frac{\tau\tanh\pi\tau}{\cosh\pi\tau} P_{-1/2+i\tau}(0) \frac{P^1_{-1/2+i\tau}(\cos\beta)}{P^2_{-1/2+i\tau}(\cos\beta_0)} \times$$

$$\times \frac{P^1_{-1/2+i\tau}(i\sinh\alpha_0) + P^1_{-1/2+i\tau}(-i\sinh\alpha_0)}{2} d\tau \qquad (0 \leqslant \beta < \beta_0) \quad (3.18)$$

In the limit when $\beta_0 = \frac{1}{2}\pi$ (problem of torsion of a space with an external circular incision and with moments $\pm M_0$ applied at the sides of this incision), the integral (3.18) can be computed in the closed form, and we have

$$\tau_{\varphi z}\,|_{z=0} = -\frac{M_0}{\pi c^3 \cosh\alpha_0 \cos\beta} \int_0^\infty \frac{\tau\tanh\pi\tau}{(1/4+\tau^2)\cosh\pi\tau} P^1_{-1/2+i\tau}(\cos\beta) \times$$

$$\times \frac{P^1_{-1/2+i\tau}(i\sinh\alpha_0) + P^1_{-1/2+i\tau}(-i\sinh\alpha_0)}{2} d\tau \qquad (0 \leqslant \beta < 1/2\,\pi) \quad (3.19)$$

If we utilise the formulas

$$\frac{P^1_{-1/2+i\tau}(i\sinh\alpha) + P^1_{-1/2+i\tau}(-i\sinh\alpha)}{2}\cosh\alpha = \quad (3.20)$$

$$= -\left(\frac{1}{4}+\tau^2\right) P_{-1/2+i\tau}(0) \int_0^\alpha \frac{P_{-1/2+i\tau}(\cosh t)}{\sqrt{\sinh^2\alpha - \sinh^2 t}} \cosh \sinh dt$$

$$\frac{1}{\sqrt{\sinh^2 t + \cos^2\beta}} = \pi \int_0^\infty \frac{\tanh\pi\tau}{\cosh\pi\tau} P_{-1/2+i\tau}(0) P_{-1/2+i\tau}(\cos\beta) P_{-1/2+i\tau}(\cosh)\, d\tau \quad (3.21)$$

then (3.19) after some transformations, yields

$$\tau_{\varphi z}\,|_{z=0} = \frac{M_0 \sinh\alpha_0 \sin\beta}{\pi^2 c^3 \cosh^2\alpha_0 \cos\beta} \frac{1}{\sinh^2\alpha_0 + \cos^2\beta} = \frac{M_0\sqrt{r_0^2 - c^2}}{\pi^2 r_0^2} \frac{r}{(r_0^2 - r^2)\sqrt{c^2 - r^2}} \quad (3.22)$$

where $r_0 = c\cosh\alpha_0$ is the radius of the circumference along which the tortional moments are applied. The possibility of obtaining solutions of a large number of problems of mathematical physics by the method outlined above, shows clearly the need for availability of the functions

$$\begin{aligned}\operatorname{Re} P_{-1/2+i\tau}(i\sinh\alpha) &= \frac{P_{-1/2+i\tau}(i\sinh\alpha)+P_{-1/2+i\tau}(-i\sinh\alpha)}{2}\\ \operatorname{Im} P_{-1/2+i\tau}(i\sinh\alpha) &= \frac{P_{-1/2+i\tau}(i\sinh\alpha)-P_{-1/2+i\tau}(-i\sinh\alpha)}{2i}\end{aligned}\qquad \begin{pmatrix}-\infty<\alpha<\infty\\ 0\leqslant\tau<\infty\end{pmatrix} \tag{3.23}$$

in the tabulated form. Solution of the last problem can be considered as a continuation of work on tabulating spherical functions with complex index $\nu = -1/2 + i\tau$, performed by the Computational Center of the Acad. of Sciences SSSR [7 and 8].

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Translated by L.K.